

Monogamy equality in $2 \otimes 2 \otimes d$ quantum systems

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There is an interesting property about multipartite entanglement, called the monogamy of entanglement. The property can be shown by the monogamy inequality, called the Coffman-Kundu-Wootters inequality [Phys. Rev. A **61**, 052306 (2000); Phys. Rev. Lett. **96**, 220503 (2006)], and more explicitly by the monogamy equality in terms of the concurrence and the concurrence of assistance, $\mathcal{C}_{A(BC)}^2 = \mathcal{C}_{AB}^2 + (\mathcal{C}_{AC}^a)^2$, in the three-qubit system. In this paper, we consider the monogamy equality in $2 \otimes 2 \otimes d$ quantum systems. We show that $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB}$ if and only if $\mathcal{C}_{AC}^a = 0$, and also show that if $\mathcal{C}_{A(BC)} = \mathcal{C}_{AC}^a$ then $\mathcal{C}_{AB} = 0$, while there exists a state in a $2 \otimes 2 \otimes d$ system such that $\mathcal{C}_{AB} = 0$ but $\mathcal{C}_{A(BC)} > \mathcal{C}_{AC}^a$.

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Entanglement provides us with a lot of useful applications in quantum communications, such as quantum key distribution and teleportation. In order to apply entanglement to more various and useful quantum information processing, there are several important things which we should take into account. One is to quantify the degree of entanglement, and another one is to know about more properties of entanglement. We here consider two measures of entanglement, and investigate some properties of entanglement related to the two entanglement measures in multipartite systems, especially $2 \otimes 2 \otimes d$ quantum systems.

Wootters' *concurrence* [1], \mathcal{C} has been considered as one of the simplest measure of entanglement, although there does not in general exist its explicit formula. For any pure state $|\phi\rangle_{AB}$, it is defined as $\mathcal{C}(|\phi\rangle_{AB}) = \sqrt{2(1 - \text{tr}\rho_A^2)}$, where $\rho_A = \text{tr}_B|\phi\rangle_{AB}\langle\phi|$. Note that $\sqrt{2(1 - \text{tr}\rho_A^2)} = 2\sqrt{\det\rho_A}$ in $2 \otimes d$ systems. For any mixed state ρ_{AB} , it is defined as

$$\mathcal{C}(\rho_{AB}) = \min \sum_k p_k \mathcal{C}(|\phi_k\rangle_{AB}), \quad (1)$$

where the minimum is taken over its all possible decompositions, $\rho_{AB} = \sum_k p_k |\phi_k\rangle_{AB}\langle\phi_k|$. Recently, another measure of entanglement has been presented, and it is called the *concurrence of assistance* (CoA) [2], which is defined as

$$\mathcal{C}^a(\rho_{AB}) = \max \sum_k p_k \mathcal{C}(|\phi_k\rangle_{AB}), \quad (2)$$

where the maximum is taken over all possible decompositions of ρ_{AB} .

In multiqubit systems, there is an interesting property about multipartite entanglement, which is called the *monogamy of entanglement* (MoE). Coffman, Kundu, and Wootters (CKW) first proposed the monogamy inequality [3], which states the MoE in the 3-qubit system,

$$\mathcal{C}_{A(BC)}^2 \geq \mathcal{C}_{AB}^2 + \mathcal{C}_{AC}^2, \quad (3)$$

and then its generalization was proved by Osborne and Verstraete [4]. Symmetrically, its dual inequality in terms of the CoA for 3-qubit states,

$$\mathcal{C}_{A(BC)}^2 \leq (\mathcal{C}_{AB}^a)^2 + (\mathcal{C}_{AC}^a)^2, \quad (4)$$

and its generalization into n -qubit cases have been also shown in [5, 6].

In particular, for 3-qubit states, it can be readily proved that the monogamy equality [7, 8],

$$\mathcal{C}_{A(BC)}^2 = \mathcal{C}_{AB}^2 + (\mathcal{C}_{AC}^a)^2 \quad (5)$$

holds. We note that this monogamy equality shows the MoE more explicitly than the CKW inequality. Thus, it could be important to investigate whether the monogamy equality would be possible in any higher dimensional tripartite quantum systems, and could be helpful for us to understand multipartite entanglement.

In this paper, we consider the monogamy equality in $2 \otimes 2 \otimes d$ systems. We show that $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB}$ if and only if $\mathcal{C}_{AC}^a = 0$, and also show that if $\mathcal{C}_{A(BC)} = \mathcal{C}_{AC}^a$ then $\mathcal{C}_{AB} = 0$, whereas there exists a state in a $2 \otimes 2 \otimes d$ system such that $\mathcal{C}_{AB} = 0$ but $\mathcal{C}_{A(BC)} > \mathcal{C}_{AC}^a$.

Now, we present the first main theorem.

Theorem 1. *Let $|\Psi\rangle_{ABC}$ be a state in a $2 \otimes 2 \otimes d$ system. Then the followings are equivalent.*

(i) $|\Psi\rangle$ is of the form $|\phi\rangle_A \otimes |\psi\rangle_{BC}$ or $|\phi'\rangle_C \otimes |\psi'\rangle_{AB}$.

(ii) $\mathcal{C}_{AC}^a = 0$.

(iii) $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB}$.

In order to prove Theorem 1, we introduce the following lemma, which is called the *Lewenstein-Sanpera decomposition* for two-qubit states [9].

Lemma 2. *Let ρ be a density matrix on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Then ρ has a unique decomposition in the form $\rho = \lambda\rho_s + (1 - \lambda)P_e$, where ρ_s is a separable density matrix, $P_e = |\Psi_e\rangle\langle\Psi_e|$ for a pure entangled state $|\Psi_e\rangle$, and $\lambda \in [0, 1]$ is maximal.*

We now give the proof of the first main theorem.

Proof of Theorem 1. We first prove that (i) is equivalent to (ii). Since ρ_{AC} is in the form of $|\psi\rangle_A\langle\psi| \otimes \sigma_C$ or $|\psi\rangle_C\langle\psi| \otimes \sigma_A$, it is trivial that $\mathcal{C}_{AC}^a = 0$. Conversely, suppose that ρ_{AC} is not in the form of $|\psi\rangle_A\langle\psi| \otimes \sigma_C$ or $|\psi\rangle_C\langle\psi| \otimes \sigma_A$. Then ρ_A and ρ_C have at least rank 2. Since $\mathcal{C}_{AC}^a = 0$,

$$\rho_{AC} = \sum_i p_i |\phi_i\rangle_A \langle\phi_i| \otimes |\psi_i\rangle_C \langle\psi_i|, \quad (6)$$

and there exists at least one pair (i, j) such that $|\langle\phi_i|\phi_j\rangle| \neq 1$ and $|\langle\psi_i|\psi_j\rangle| \neq 1$. By Hughston-Jozsa-Wootters (HJW) theorem [10], $\rho_{AC} = \sum_k q_k |\Phi_k\rangle_{AC} \langle\Phi_k|$ such that at least one

$$|\Phi_k\rangle_{AC} = \alpha |\phi_i\rangle_A |\psi_i\rangle_C + \beta |\phi_j\rangle_A |\psi_j\rangle_C \quad (7)$$

is entangled ($\alpha \neq 0$ and $\beta \neq 0$), and hence $\mathcal{C}_{AC}^a > 0$.

Since if $|\Psi\rangle_{ABC}$ is in the form $|\phi\rangle_A \otimes |\psi\rangle_{BC}$ or $|\phi'\rangle_C \otimes |\psi'\rangle_{AB}$ then it is clear that $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB}$, the final one for completing the proof of this theorem, is to show that if $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB}$ then $|\Psi\rangle_{ABC}$ has the form $|\phi\rangle_A \otimes |\psi\rangle_{BC}$ or $|\phi'\rangle_C \otimes |\psi'\rangle_{AB}$.

We assume that $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB} \neq 0$ (If $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB} = 0$ then $|\Psi\rangle_{ABC}$ is of the form $|\phi\rangle_A \otimes |\psi\rangle_{BC}$, and so this theorem is trivially true). Then we clearly have $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB}^a = \mathcal{C}_{AB}$, that is, the average concurrence of any decomposition of ρ_{AB} is equal to $\mathcal{C}_{A(BC)}$. By Lemma 2, $\rho_{AB} = \lambda \rho_s + (1 - \lambda) P_e$, where ρ_s is separable and P_e is purely entangled. Then since $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB} = (1 - \lambda) \mathcal{C}_{AB}(P_e)$, we can see that ρ_s is in the form of $|0\rangle_A \langle 0| \otimes \sigma_B$ or $(x|0\rangle_A \langle 0| + y|1\rangle_A \langle 1|) \otimes |0\rangle_B \langle 0|$ up to local unitary operations.

We now let

$$\begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \equiv (1 - \lambda) \text{tr}_B(P_e). \quad (8)$$

Then $(1 - \lambda) \mathcal{C}_{AB}(P_e) = 2\sqrt{ac - |b|^2}$, and

$$\rho_A = \begin{pmatrix} \lambda + a & b \\ b^* & c \end{pmatrix} \text{ or } \begin{pmatrix} \lambda x + a & b \\ b^* & \lambda y + c \end{pmatrix}. \quad (9)$$

Thus, since $\mathcal{C}_{A(BC)} = 2\sqrt{(\lambda + a)c - |b|^2}$ or $2\sqrt{(\lambda x + a)(\lambda y + c) - |b|^2}$, it is obtained that $\lambda = 0$, that is, $\rho_{AB} = P_e$. Therefore, we can conclude that $|\Psi\rangle_{ABC}$ is of the form $|\phi\rangle_C \otimes |\psi\rangle_{AB}$. \square

We now present the second main theorem.

Theorem 3. If $\mathcal{C}_{A(BC)} = \mathcal{C}_{AC}^a$ then $\mathcal{C}_{AB} = 0$.

For the proof of Theorem 3, we introduce the two following lemmas. One is as follows.

Lemma 4. If ρ and σ are 2×2 positive matrices with $\text{rank}(\rho) = 1$ and $\text{rank}(\sigma) = 2$, respectively then for any $\lambda_j \geq 0$, $\sqrt{\det(\lambda_0 \rho + \lambda_1 \sigma)} \geq \lambda_1 \sqrt{\det \sigma}$, where the equality holds if and only if $\lambda_0 = 0$ or $\lambda_1 = 0$. If ρ and σ_j are 2×2

positive matrices with $\text{rank}(\rho) = 1$ and $\text{rank}(\sigma_j) = 2$, respectively then for any $\alpha, \beta_j \geq 0$,

$$\sqrt{\det(\alpha \rho + \beta_0 \sigma_0 + \beta_1 \sigma_1)} \geq \sqrt{\det(\beta_0 \sigma_0 + \beta_1 \sigma_1)}, \quad (10)$$

where the equality holds if and only if $\alpha = 0$ or $\beta_j = 0$.

Proof. To begin with, we show the first statement. Without loss of generality, we may assume that $\rho = |\psi\rangle\langle\psi|$ and $\sigma = a|0\rangle\langle 0| + b|1\rangle\langle 1|$, where $|\psi\rangle = x|0\rangle + y|1\rangle$ and $a, b > 0$. Then

$$\begin{aligned} \sqrt{\det(\lambda_0 \rho + \lambda_1 \sigma)} &= \sqrt{\lambda_0 \lambda_1 (|x|^2 b + |y|^2 a) + \lambda_1^2 ab} \\ &\geq \sqrt{\lambda_1^2 ab} \\ &= \lambda_1 \sqrt{\det \sigma}. \end{aligned} \quad (11)$$

It is clear that the equality in (11) holds if and only if $\lambda_0 = 0$ or $\lambda_1 = 0$. Similarly, we can show the second statement. \square

The other lemma is called the *Minkowski determinant inequality theorem* [11].

Lemma 5. If $n \times n$ matrices A, B are positive definite, then

$$[\det(A + B)]^{1/n} \geq (\det A)^{1/n} + (\det B)^{1/n}. \quad (12)$$

The equality in (12) holds if and only if $B = cA$ for some $c \geq 0$.

In the proof of the second main theorem, we will use Lemma 5 just in the case of $n = 2$.

Proof of Theorem 3. We first let

$$\rho_{AC} = \sum_{i \in I} \lambda_i |\psi_i\rangle_{AC} \langle\psi_i| \quad (13)$$

be an optimal decomposition of ρ_{AC} for the CoA, \mathcal{C}_{AC}^a . Then we can consider the three cases according to the rank of $\text{tr}_C(|\psi_i\rangle_{AC} \langle\psi_i|)$: (i) $\text{rank}[\text{tr}_C(|\psi_i\rangle_{AC} \langle\psi_i|)] = 1$ for all $i \in I$, (ii) there exist two nonempty subsets I_1 and $I_2 = I - I_1$ of I such that $\text{rank}[\text{tr}_C(|\psi_i\rangle_{AC} \langle\psi_i|)] = 1$ for all $i \in I_1$ and $\text{rank}[\text{tr}_C(|\psi_j\rangle_{AC} \langle\psi_j|)] = 2$ for all $j \in I_2$, (iii) $\text{rank}[\text{tr}_C(|\psi_i\rangle_{AC} \langle\psi_i|)] = 2$ for all $i \in I$.

We now prove this theorem case by case.

(Case i) Assume that $\text{rank}[\text{tr}_C(|\psi_i\rangle_{AC} \langle\psi_i|)] = 1$ for all $i \in I$. Then since $|\psi_i\rangle_{AC} \langle\psi_i|$'s are all pure and separable states, $\mathcal{C}_{AC}^a = 0$ and so $\mathcal{C}_{A(BC)} = 0$. Thus, we have $|\Psi\rangle_{ABC}$ is of the form $|\phi\rangle_A \otimes |\psi\rangle_{BC}$, and it immediately follows that $\mathcal{C}_{AB} = 0$.

(Case ii) Assume that there exist two nonempty subsets I_1 and $I_2 = I - I_1$ of I such that $\text{rank}[\text{tr}_C(|\psi_i\rangle_{AC} \langle\psi_i|)] = 1$ for all $i \in I_1$ and $\text{rank}[\text{tr}_C(|\psi_j\rangle_{AC} \langle\psi_j|)] = 2$ for all $j \in I_2$. The by Lemma 4 and Lemma 5, we can obtain the following inequality.

$$\begin{aligned}
\mathcal{C}_{A(BC)} &= 2 \sqrt{\det \left[\sum_{i \in I_1} \lambda_i \text{tr}_C(|\psi_i\rangle_{AC}\langle\psi_i|) + \sum_{j \in I_2} \lambda_j \text{tr}_C(|\psi_j\rangle_{AC}\langle\psi_j|) \right]} \geq 2 \sqrt{\det \left[\sum_{j \in I_2} \lambda_j \text{tr}_C(|\psi_j\rangle_{AC}\langle\psi_j|) \right]} \\
&\geq 2 \sum_{j \in I_2} \lambda_j \sqrt{\det(\text{tr}_C(|\psi_j\rangle_{AC}\langle\psi_j|))} = \mathcal{C}_{AC}^a.
\end{aligned} \tag{14}$$

Since $\mathcal{C}_{A(BC)} = \mathcal{C}_{AC}^a$, the equality in the first inequality should hold, and hence $\lambda_i = 0$ for all $i \in I_1$ or $\lambda_j = 0$ for all $j \in I_2$ by Lemma 4. This means that it is sufficient to consider the cases (i) and (iii).

(Case iii) We assume that $\text{rank}[\text{tr}_C(|\psi_i\rangle_{AC}\langle\psi_i|)] = 2$ for all $i \in I$. Then by Lemma 5,

$$\begin{aligned}
\mathcal{C}_{A(BC)} &= 2 \sqrt{\det \left[\text{tr}_C \sum_{i \in I} \lambda_i |\psi_i\rangle_{AC}\langle\psi_i| \right]} \\
&\geq 2 \sum_{i \in I} \lambda_i \sqrt{\det(\text{tr}_C(|\psi_i\rangle_{AC}\langle\psi_i|))} \\
&= \mathcal{C}_{AC}^a,
\end{aligned} \tag{15}$$

and the equality in the inequality (15) holds if and only if $\text{tr}_C(|\psi_i\rangle_{AC}\langle\psi_i|) = \rho_A$ for all $i \in I$.

Let $\rho_A = \mu_0|0\rangle_A\langle 0| + \mu_1|1\rangle_A\langle 1|$ be its spectral decomposition. By the Gisin-Hughston-Jozsa-Wootters theorem [10, 12], for $0, 1 \in I$, there is a unitary operator U such that

$$\begin{aligned}
|\psi_0\rangle_{AC} &= \sqrt{\mu_0}|0\rangle_A|0\rangle_C + \sqrt{\mu_1}|1\rangle_A|1\rangle_C, \\
|\psi_1\rangle_{AC} &= \sqrt{\mu_0}|0\rangle_A U|0\rangle_C + \sqrt{\mu_1}|1\rangle_A U|1\rangle_C.
\end{aligned} \tag{16}$$

Let $\rho_{AC} = \nu_0|\phi_0\rangle_{AC}\langle\phi_0| + \nu_1|\phi_1\rangle_{AC}\langle\phi_1|$ be the spectral decomposition of ρ_{AC} . Then since $\text{rank}(\rho_{AC}) = 2$, the eigenvectors $|\tilde{\phi}_0\rangle = \sqrt{\nu_0}|\phi_0\rangle$ and $|\tilde{\phi}_1\rangle = \sqrt{\nu_1}|\phi_1\rangle$ should be linear combinations of $|\psi_0\rangle$ and $|\psi_1\rangle$. It follows that

$$|\Psi\rangle_{ABC} = |\tilde{\phi}_0\rangle_{AC}|0\rangle_B + |\tilde{\phi}_1\rangle_{AC}|1\rangle_B, \tag{17}$$

where $|\tilde{\phi}_0\rangle = x_0|\psi_0\rangle + x_1|\psi_1\rangle$ and $|\tilde{\phi}_1\rangle = y_0|\psi_0\rangle + y_1|\psi_1\rangle$. Let

$$|\Psi'\rangle_{ABC} = |\tilde{\phi}_0'\rangle_{AC}|0\rangle_B + |\tilde{\phi}_1'\rangle_{AC}|1\rangle_B, \tag{18}$$

where $|\tilde{\phi}_0'\rangle = x_1^*|\psi_0\rangle + x_0^*|\psi_1\rangle$ and $|\tilde{\phi}_1'\rangle = y_1^*|\psi_0\rangle + y_0^*|\psi_1\rangle$. Then, by tedious but straightforward calculations, we can check that the partial transpose $\rho_{AB}^{T_B}$ of ρ_{AB} is equal to $\rho'_{AB} = \text{tr}_C(|\Psi'\rangle_{ABC}\langle\Psi'|)$, and thus ρ_{AB} has positive partial transposition (PPT). Therefore, $\mathcal{C}_{AB} = 0$. \square

So far, we have seen the case that the monogamy equality holds in $2 \otimes 2 \otimes d$ systems. We now exhibit a counterexample that the monogamy equality does not hold, in particular, $\mathcal{C}_{AB} = 0$ but $\mathcal{C}_{A(BC)} > \mathcal{C}_{AC}^a$.

Example 6. Consider two orthogonal states in the $2 \otimes 3$ quantum system, $|x\rangle = (|02\rangle + \sqrt{2}|10\rangle)/\sqrt{3}$, $|y\rangle = (|12\rangle + \sqrt{2}|01\rangle)/\sqrt{3}$. We now take into account the following state in the $2 \otimes 2 \otimes 3$ quantum system,

$$\begin{aligned}
|\Psi\rangle_{ABC} &= \frac{1}{\sqrt{2}}|x\rangle_{AC}|0\rangle_B + \frac{1}{\sqrt{2}}|y\rangle_{AC}|1\rangle_B \\
&= \frac{1}{\sqrt{6}}|002\rangle_{ABC} + \frac{1}{\sqrt{3}}|100\rangle_{ABC} \\
&\quad + \frac{1}{\sqrt{6}}|112\rangle_{ABC} + \frac{1}{\sqrt{3}}|011\rangle_{ABC}.
\end{aligned} \tag{19}$$

Then since $\rho_A = (|0\rangle_A\langle 0| + |1\rangle_A\langle 1|)/2$, it is clear that $\mathcal{C}_{A(BC)} = 1$, and since $\rho_{AC} = (|x\rangle_{AC}\langle x| + |y\rangle_{AC}\langle y|)/2$, by the HJW theorem, for any decompositions $\rho_{AC} = \sum_i p_i |\phi_i\rangle_{AC}\langle\phi_i|$, $\sqrt{p_i}|\phi_i\rangle_{AC} = (c_{i1}|x\rangle_{AC} + c_{i2}|y\rangle_{AC})/\sqrt{2}$ for some unitary operator (c_{ij}) with $2p_i = |c_{i1}|^2 + |c_{i2}|^2$. Then

$$2p_i \text{tr}_C(|\phi_i\rangle_{AC}\langle\phi_i|) = \frac{1}{3} \begin{pmatrix} |c_{i1}|^2 + 2|c_{i2}|^2 & c_{i1}c_{i2}^* \\ c_{i2}c_{i1}^* & |c_{i2}|^2 + 2|c_{i1}|^2 \end{pmatrix}, \tag{20}$$

and hence

$$\text{tr}_C(|\phi_i\rangle_{AC}\langle\phi_i|) = \frac{1}{3}I_A + \frac{1}{3}|\psi_i\rangle_A\langle\psi_i| \tag{21}$$

with $|\psi_i\rangle = (c_{i2}^*|0\rangle + c_{i1}^*|1\rangle)/\sqrt{2p_i}$. Thus we obtain that $\mathcal{C}_{AC} = \frac{2\sqrt{2}}{3} = \mathcal{C}_{AC}^a$. Since

$$\rho_{AB} = \frac{1}{6} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

clearly has PPT, $\mathcal{C}_{AB} = 0$. Therefore, there exists a quantum state in the $2 \otimes 2 \otimes 3$ system such that $\mathcal{C}_{AB} = 0$, but $\mathcal{C}_{A(BC)} = 1 > \frac{2\sqrt{2}}{3} = \mathcal{C}_{AC}^a$.

In conclusion, we have considered the monogamy equality in $2 \otimes 2 \otimes d$ quantum systems. We have shown that $\mathcal{C}_{A(BC)} = \mathcal{C}_{AB}$ if and only if $\mathcal{C}_{AC}^a = 0$, and have also shown that if $\mathcal{C}_{A(BC)} = \mathcal{C}_{AC}^a$ then $\mathcal{C}_{AB} = 0$, while there exists a state in a $2 \otimes 2 \otimes d$ system such that $\mathcal{C}_{AB} = 0$ but $\mathcal{C}_{A(BC)} > \mathcal{C}_{AC}^a$. However, in $2 \otimes 2 \otimes d$ quantum systems, the monogamy inequality in terms of the concurrence and the CoA, $\mathcal{C}_{A(BC)}^2 \geq \mathcal{C}_{AB}^2 + (\mathcal{C}_{AC}^a)^2$, has been still unknown.

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